THE STURM-LIOUVILLE PROBLEM AND THE POLAR REPRESENTATION THEOREM

JORGE REZENDE

Grupo de Física-Matemática da Universidade de Lisboa
Av. Prof. Gama Pinto 2, 1649-003 Lisboa, PORTUGAL
and
Departamento de Matemática,
Faculdade de Ciências da Universidade de Lisboa
e-mail: rezende@cii.fc.ul.pt

Dedicated to the memory of Professor Ruy Luís Gomes

Abstract: The polar representation theorem for the n-dimensional time-dependent linear Hamiltonian system

$$\dot{Q} = BQ + CP, \ \ \dot{P} = -AQ - B^*P,$$

with continuous coefficients, states that, given two isotropic solutions (Q_1, P_1) and (Q_2, P_2), with the identity matrix as Wronskian, the formula

$$Q_2 = r \cos \varphi, \ \ Q_1 = r \sin \varphi,$$

holds, where r and φ are continuous matrices, $\det r \neq 0$ and φ is symmetric.

In this article we use the monotonicity properties of the matrix φ eigenvalues in order to obtain results on the Sturm-Liouville problem.

AMS Subj. Classification: 34B24, 34C10, 34A30
Key words: Sturm-Liouville theory, Hamiltonian systems, polar representation.

1. Introduction

Let $n = 1, 2, \ldots$. In this article, $(.,.)$ denotes the natural inner product in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ one writes $x^2 = (x, x)$, $|x| = (x, x)^{\frac{1}{2}}$. If

M is a real matrix, we shall denote M^* its transpose. M_{jk} denotes the matrix entry located in row j and column k. I_n is the identity $n \times n$ matrix. M_{jk} can be a matrix. For example, M can have the four blocks M_{11}, M_{12}, M_{21}, M_{22}. In a case like this one, if $M_{12} = M_{21} = 0$, we write $M = \operatorname{diag}(M_{11}, M_{22})$.

1.1. The symplectic group and the polar representation theorem.

Consider the time-dependent linear Hamiltonian system

$$\dot{Q} = BQ + CP,\ \dot{P} = -AQ - B^*P, \tag{1.1}$$

where A, B and C are time-dependent $n \times n$ matrices. A and C are symmetric. The dot means time derivative, the derivative with respect to τ. The time variable τ belongs to an interval. Without loss of generality we shall assume that this interval is $[0, T[$, $T > 0$. T can be ∞. In the following t, $0 < t < T$, is also a time variable and $\tau \in [0, t]$.

If (Q_1, P_1) and (Q_2, P_2) are solutions of (1.1) one denotes the Wronskian (which is constant) by

$$W(Q_1, P_1; Q_2, P_2) \equiv W = P_1^* Q_2 - Q_1^* P_2.$$

A solution (Q, P) of (1.1) is called isotropic if $W(Q, P; Q, P) = 0$. From now on (Q_1, P_1) and (Q_2, P_2) will denote two isotropic solutions of (1.1) such that $W(Q_1, P_1; Q_2, P_2) = I_n$. This means that

$$P_1^* Q_2 - Q_1^* P_2 = I_n\ ,\ P_1^* Q_1 = Q_1^* P_1\ ,\ P_2^* Q_2 = Q_2^* P_2.$$

These relations express precisely that, for each $\tau \in [0, T[$ the $2n \times 2n$ matrix

$$\Phi = \begin{bmatrix} Q_2 & Q_1 \\ P_2 & P_1 \end{bmatrix} \tag{1.2}$$

is symplectic. Its left inverse and, therefore, its inverse, is given by

$$\Phi^{-1} = \begin{bmatrix} P_1^* & -Q_1^* \\ -P_2^* & Q_2^* \end{bmatrix}.$$

As it is well-known the $2n \times 2n$ symplectic matrices form a group, the symplectic group.

Then, one has

$$P_1 Q_2^* - P_2 Q_1^* = I_n, \quad Q_1 Q_2^* = Q_2 Q_1^*, \quad P_1 P_2^* = P_2 P_1^*,$$

and, therefore,

$$Q_2^* P_1 - P_2^* Q_1 = I_n, \quad Q_2 P_1^* - Q_1 P_2^* = I_n,$$

and the following matrices, whenever they make sense, are symmetric

$$P_2Q_2^{-1},\quad Q_1P_1^{-1},\quad Q_2P_2^{-1},\quad P_1Q_1^{-1},$$
$$Q_2^{-1}Q_1,\quad P_2^{-1}P_1,\quad Q_1^{-1}Q_2,\quad P_1^{-1}P_2.$$

Denote by J, S and M, the following $2n\times 2n$ matrices

$$J=\begin{bmatrix}0 & -I_n\\ I_n & 0\end{bmatrix},\quad S=\begin{bmatrix}A & B^*\\ B & C\end{bmatrix},$$

and $M=-JS$. J is symplectic and S is symmetric.

One says that the $2n\times 2n$ matrix L is antisymplectic if $LJL^*=-J$. Notice that the product of two antisymplectic matrices is symplectic, and that the product of an antisymplectic matrix by a symplectic one is antisymplectic. We shall use this definition later.

Notice that if $n=1$ and L is a 2×2 matrix, then one has $LJL^*=(\det L)\,J$.

Equation (1.1) can then be written

$$\dot{\Phi}=M\Phi.$$

Notice that, if Φ is symplectic, Φ^* is symplectic, and

$$\Phi^{-1}=-J\Phi^*J,\ \ \Phi^*J\Phi=J,\ \ \Phi J\Phi^*=J.$$

When we have a C^1 function $\tau\longmapsto\Phi(\tau)$, $\dot{\Phi}J\Phi^*+\Phi J\dot{\Phi}^*=0$. Hence, $\dot{\Phi}J\Phi^*$ is symmetric and one can recover M:

$$M=\dot{\Phi}\Phi^{-1}=-\dot{\Phi}J\Phi^*J.$$

This means that from Φ one can obtain A, B, and C:

$$A=\dot{P}_1P_2^*-\dot{P}_2P_1^*,\quad C=\dot{Q}_1Q_2^*-\dot{Q}_2Q_1^*,$$
$$B=-\dot{Q}_1P_2^*+\dot{Q}_2P_1^*=Q_1\dot{P}_2^*-Q_2\dot{P}_1^*.$$

The proof of the following theorem on a polar representation can be found in [3]. See also [4], [5].

Theorem 1.1. *Assume that $C(\tau)$ is always >0 (or always <0) and of class C^1. Consider two isotropic solutions of (1.1), (Q_1,P_1) and (Q_2,P_2), such that $W=I_n$. Then, there are C^1 matrix-valued functions $r(\tau)$, $\varphi(\tau)$, for $\tau\in[0,T[$, such that: a) $\det r(\tau)\neq 0$ and $\varphi(\tau)$ is symmetric for every τ; b) the eigenvalues of φ are C^1 functions of τ, with strictly positive (negative) derivatives; c) one has*

$$Q_2(\tau)=r(\tau)\cos\varphi(\tau)\quad\text{and}\quad Q_1(\tau)=r(\tau)\sin\varphi(\tau).$$

Notice that φ is not unique and that

$$\frac{d}{d\tau} Q_2^{-1} Q_1 = Q_2^{-1} C Q_2^{*-1}, \tag{1.3}$$

whenever $\det Q_2(\tau) \neq 0$ (see [3]).

Example 1.1. Consider $n = 1$, $B = 0$, $A = C = 1$. Let $k_1, k_2 \in \mathbb{R}$. For $k_2 > 0$, let

$$Q_2(\tau) = k_2^{-1/2} \cos\tau, \;\; Q_1(\tau) = k_2^{-1/2}(k_1 \cos\tau + k_2 \sin\tau).$$

Then there exists an increasing continuous function of τ, $\xi(k_1, k_2, \tau) \equiv \xi(\tau)$, $\tau \in \mathbb{R}$, such that

$$Q_2(\tau) = r(\tau)\cos\xi(\tau), \;\; Q_1(\tau) = r(\tau)\sin\xi(\tau),$$

where $r(\tau) = k_2^{-1/2}\sqrt{\cos^2\tau + (k_1\cos\tau + k_2\sin\tau)^2}$. The function ξ is not unique in the sense that two such functions differ by $2k\pi$, $k \in \mathbb{Z}$. For $\tau \neq \frac{\pi}{2} + k\pi$, one has

$$k_1 + k_2 \tan\tau = \tan\xi(\tau). \tag{1.4}$$

This formula shows that $\lim_{\tau\to\pm\infty} \xi(\tau) = \pm\infty$.

For $k_2 < 0$, one defines, obviously,

$$\xi(k_1, k_2, \tau) = -\xi(-k_1, -k_2, \tau).$$

When $k_2 = 0$, ξ is a constant function. For every $k_2 \in \mathbb{R}$, formula (1.4) remains valid.

One can fix ξ by imposing $-\frac{\pi}{2} < \xi(0) < \frac{\pi}{2}$, as we shall do from now on.

For $k_2 > 0$, one has $\xi\left(\frac{\pi}{2} + k\pi\right) = \frac{\pi}{2} + k\pi$, and for $k_2 < 0$, one has $\xi\left(\frac{\pi}{2} + k\pi\right) = -\frac{\pi}{2} - k\pi$, for every $k \in \mathbb{Z}$.

If S is a symmetric $n \times n$ matrix, and Ω is an orthogonal matrix that diagonalizes S, $S = \Omega\,\mathrm{diag}(s_1, s_2, \ldots, s_n)\Omega^*$, we denote

$$\xi(k_1, k_2, S) \equiv \xi(S) = \Omega\,\mathrm{diag}(\xi(s_1), \xi(s_2), \ldots, \xi(s_n))\Omega^*.$$

Define now

$$\zeta(\tau) \equiv \zeta(k_1, k_2, \tau) = -\xi(k_1, k_2, \tau) + \frac{\pi}{2}. \tag{1.5}$$

Then $0 < \zeta(0) < \pi$, and

$$(k_1 + k_2\tan\tau)^{-1} = \tan\zeta(\tau),$$

for every τ such that $k_1 + k_2 \tan\tau \neq 0$.

For $k_2 > 0$, one has $\zeta\left(\frac{\pi}{2} + k\pi\right) = -k\pi$, and for $k_2 < 0$, one has $\zeta\left(\frac{\pi}{2} + k\pi\right) = (k+1)\pi$, for every $k \in \mathbb{Z}$. The function ζ is increasing for $k_2 < 0$, decreasing for $k_2 > 0$ and constant for $k_2 = 0$.

If S is a symmetric $n \times n$ matrix, one can define $\zeta(k_1, k_2, S)$ as we did before for ξ.

We shall need these functions later.

Theorem 1.1 can be extended in the following way:

Theorem 1.2. *Assume that $C(\tau)$ is of class C^1. Consider two isotropic solutions of* (1.1)*, (Q_1, P_1) and (Q_2, P_2), such that $W = I_n$. Then, there are C^1 matrix-valued functions $r(\tau)$, $\varphi(\tau)$, for $\tau \in [0, t]$, such that: a) $\det r(\tau) \neq 0$ and $\varphi(\tau)$ is symmetric for every τ; b) the eigenvalues of φ are C^1 functions of τ; c) one has*

$$Q_2(\tau) = r(\tau)\cos\varphi(\tau) \quad \text{and} \quad Q_1(\tau) = r(\tau)\sin\varphi(\tau).$$

Proof. Let us first notice that $Q_2 Q_2^* + Q_1 Q_1^* > 0$. This is proved noticing that, as $P_1 Q_2^* - P_2 Q_1^* = I_n$, one has $(P_1^* x, Q_2^* x) - (P_2^* x, Q_1^* x) = |x|^2$, which implies that $\ker Q_1^* \cap \ker Q_2^* = \{0\}$. Hence, $(Q_2^* x, Q_2^* x) + (Q_1^* x, Q_1^* x) > 0$, for every $x \neq 0$.

Define now

$$\Phi = \begin{bmatrix} Q_2 & Q_1 \\ P_2 & P_1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} \cos(k\tau) I_n & \sin(k\tau) I_n \\ -\sin(k\tau) I_n & \cos(k\tau) I_n \end{bmatrix},$$

M as before, $\Phi_1 = \Phi\Psi$ and $M_1 = \dot{\Phi}_1 \Phi_1^{-1}$. The constant k is > 0. Then, one has

$$M_1 = M + \Phi\dot{\Psi}\Psi^{-1}\Phi^{-1}.$$

Let the $n \times n$ matrices, that are associated with M_1, be A_1, B_1 and C_1. Then

$$C_1 = C + k(Q_2 Q_2^* + Q_1 Q_1^*).$$

Hence, as $Q_2 Q_2^* + Q_1 Q_1^* > 0$, for k large enough, we have that $C_1(\tau) > 0$, for every $\tau \in [0, t]$. We can then apply Theorem 1.1. There are C^1 matrix-valued functions $r_1(\tau)$, $\varphi_1(\tau)$, for $\tau \in [0, t]$, such that

$$\begin{aligned} \cos(k\tau) Q_2(\tau) - \sin(k\tau) Q_1(\tau) &= r_1(\tau)\cos\varphi_1(\tau) \\ \sin(k\tau) Q_2(\tau) + \cos(k\tau) Q_1(\tau) &= r_1(\tau)\sin\varphi_1(\tau). \end{aligned}$$

From this, we have

$$\begin{aligned} Q_2(\tau) &= r_1(\tau)\cos(\varphi_1(\tau) - k\tau I_n) \\ Q_1(\tau) &= r_1(\tau)\sin(\varphi_1(\tau) - k\tau I_n). \end{aligned}$$

□

The generic differential equations for r and φ are easily derived from equations (15), (17) and (18) in [3].

Consider (r_0, s), with s symmetric, such that

$$\dot{r}_0 = Br_0 + Cr_0^{*-1}s, \;\; \dot{s} = sr_0^{-1}Cr_0^{*-1}s + r_0^{-1}Cr_0^{*-1} - r_0^*Ar_0.$$

Then r is of the form $r = r_0\Omega$, where Ω is any orthogonal, $\Omega^{-1} = \Omega^*$, and time-dependent C^1 matrix. From this one can derive a differential equation for rr^*.

The function φ verifies the equations

$$\frac{\cos \mathcal{C}_\varphi - I}{\mathcal{C}_\varphi}\dot{\varphi} = -\Omega^*\dot{\Omega}, \quad \frac{\sin \mathcal{C}_\varphi}{\mathcal{C}_\varphi}\dot{\varphi} = r^{-1}Cr^{*-1}, \tag{1.6}$$

where $\mathcal{C}_\varphi\dot{\varphi} = [\varphi, \dot{\varphi}] = \varphi\dot{\varphi} - \dot{\varphi}\varphi$, $(\mathcal{C}_\varphi)^2\,\dot{\varphi} \equiv \mathcal{C}_\varphi^2\dot{\varphi} = [\varphi, [\varphi, \dot{\varphi}]]$, and so on.

As in Theorem 1.1, φ is not unique. Notice that $r(\tau) = r_1(\tau)$ and $\varphi(\tau) = \varphi_1(\tau) - k\tau I_n$, with k large enough and φ_1 such that its eigenvalues are C^1 functions of τ, with strictly positive derivatives.

Remark 1.1. If one considers Φ^* instead of Φ, then Q_2 is replaced by Q_2^* and Q_1 is replaced by P_1^*. Then Theorem 1.2 gives

$$Q_2^*(\tau) = r(\tau)\cos\varphi(\tau) \quad \text{and} \quad P_2^*(\tau) = r(\tau)\sin\varphi(\tau),$$

or

$$Q_2(\tau) = \cos\varphi(\tau)\,r^*(\tau) \quad \text{and} \quad P_2(\tau) = \sin\varphi(\tau)\,r^*(\tau).$$

In this case the matrix $\varphi(\tau)$ is a generalization of the so-called Prüfer angle [1].

Denote (Q_c, P_c), (Q_s, P_s) the (isotropic) solutions of (1.1) such that

$$Q_c(0) = P_s(0) = I_n, \quad Q_s(0) = P_c(0) = 0.$$

From now on we shall denote by Φ_0 the symplectic matrix

$$\Phi_0 = \begin{bmatrix} Q_c & Q_s \\ P_c & P_s \end{bmatrix}.$$

Then $\dot{\Phi}_0 = M\Phi_0$ and $\Phi_0(0) = I_{2n}$.

1.2. The Sturm-Liouville problem.

Let $t \in [0, T[$ and $\lambda \in]l_{-1}, l_1[\subset \mathbb{R}$. The interval $]l_{-1}, l_1[$ can be as general as possible. In this article, t is the "time" variable and λ is the "eigenvalue" variable.

Consider A_0, B_0 and C_0 time and eigenvalue dependent $n\times n$ matrices. As in (1.1) A_0 and C_0 are symmetric. Define also M_0, S_0 and Φ_0 (here, $\dot{\Phi}_0 = M_0\Phi_0$) as before.

From now on we shall use the notations $A_0 \equiv A_0(\tau) \equiv A_0(\tau,\lambda)$, and the same for the other matrices.

Consider also α_j, β_j, γ_j and δ_j, $j=0,1$, eight eigenvalue dependent $n\times n$ matrices, and the problem of finding a λ and a solution

$$\tau \longmapsto (q(\tau,\lambda), p(\tau,\lambda)) \equiv (q(\tau), p(\tau)) \equiv (q,p),$$

$(q,p)\in\mathbb{R}^n\times\mathbb{R}^n$, for $\tau\in[0,t]$, $\lambda\in\,]l_{-1},l_1[$, of the system

$$\dot{q} = B_0 q + C_0 p, \quad \dot{p} = -A_0 q - B_0^* p,$$

with the "boundary" conditions

$$\begin{bmatrix}\beta_0 & \delta_0\\ \beta_1 & \delta_1\end{bmatrix}\begin{bmatrix}-q(0)\\ q(t)\end{bmatrix} + \begin{bmatrix}-\alpha_0 & \gamma_0\\ -\alpha_1 & \gamma_1\end{bmatrix}\begin{bmatrix}p(0)\\ p(t)\end{bmatrix} = 0,$$

or, equivalently,

$$\begin{bmatrix}\beta_0 & \alpha_0\\ \beta_1 & \alpha_1\end{bmatrix}\begin{bmatrix}q(0)\\ p(0)\end{bmatrix} - \begin{bmatrix}\delta_0 & \gamma_0\\ \delta_1 & \gamma_1\end{bmatrix}\begin{bmatrix}q(t)\\ p(t)\end{bmatrix} = 0.$$

Denote

$$\mathcal{S}_q = \begin{bmatrix}\beta_0 & \delta_0\\ \beta_1 & \delta_1\end{bmatrix}, \quad \mathcal{S}_p = \begin{bmatrix}-\alpha_0 & \gamma_0\\ -\alpha_1 & \gamma_1\end{bmatrix}.$$

In order to preserve the self-adjointness of the problem, one has to have self-adjoint boundary conditions $\mathcal{S}_q\mathcal{S}_p^* = \mathcal{S}_p\mathcal{S}_q^*$ [2]. This means that

$$\begin{aligned}
\alpha_0\beta_0^* + \delta_0\gamma_0^* &= \beta_0\alpha_0^* + \gamma_0\delta_0^*,\\
\alpha_1\beta_1^* + \delta_1\gamma_1^* &= \beta_1\alpha_1^* + \gamma_1\delta_1^*,\\
\alpha_0\beta_1^* + \delta_0\gamma_1^* &= \beta_0\alpha_1^* + \gamma_0\delta_1^*.
\end{aligned}$$

Remark 1.2. Consider F a eigenvalue dependent symplectic matrix. If Φ is a symplectic solution of $\dot{\Phi} = M_0\Phi$, then all previous formulas involving Φ, M_0, $\mathcal{S}_q$ and $\mathcal{S}_p$ remain valid if we replace Φ by $F^{-1}\Phi$, M_0 by $F^{-1}M_0F$, $\mathcal{S}_q$ by $\mathcal{S}_q\operatorname{diag}(F_{11},F_{11})+\mathcal{S}_p\operatorname{diag}(-F_{21},F_{21})$, and $\mathcal{S}_p$ by $\mathcal{S}_q\operatorname{diag}(-F_{12},F_{12})+\mathcal{S}_p\operatorname{diag}(F_{22},F_{22})$.

As

$$\begin{bmatrix}q(\tau)\\ p(\tau)\end{bmatrix} = \Phi_0(\tau)\begin{bmatrix}q(0)\\ p(0)\end{bmatrix}$$

one obtains

$$\left(\begin{bmatrix}\beta_0 & \alpha_0\\ \beta_1 & \alpha_1\end{bmatrix} - \begin{bmatrix}\delta_0 & \gamma_0\\ \delta_1 & \gamma_1\end{bmatrix}\Phi_0\left(t\right)\right)\begin{bmatrix}q\left(0\right)\\ p\left(0\right)\end{bmatrix} = 0.$$

In order to have a non trivial solution, $\left(q\left(0\right), p\left(0\right)\right) \neq (0,0)$, of this system we must have

$$\det\left(\begin{bmatrix}\beta_0 & \alpha_0\\ \beta_1 & \alpha_1\end{bmatrix} - \begin{bmatrix}\delta_0 & \gamma_0\\ \delta_1 & \gamma_1\end{bmatrix}\Phi_0\left(t\right)\right) = 0. \tag{1.7}$$

We shall need now the following lemma.

Lemma 1.3. *Consider a, b, c and d, $n \times n$ real matrices, such that $ab^* = ba^*$ and $cd^* = dc^*$. Let*

$$N = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

Then $\det N = 0$ *if and only if* $\det\left(ad^* - bc^*\right) = 0$.

Proof. From

$$NJN^*J = \operatorname{diag}\left(-ad^* + bc^*, -da^* + cb^*\right),$$

one has $\left(\det N\right)^2 = \left(\det\left(ad^* - bc^*\right)\right)^2$. The lemma follows now easily. □

In order to apply this lemma to equation (1.7) we need to assume that, from now on,

$$\beta_j\alpha_j^* + \delta_j\gamma_j^* - \beta_j Q_s^*\left(t\right)\delta_j^* - \beta_j P_s^*\left(t\right)\gamma_j^* - \delta_j Q_c\left(t\right)\alpha_j^* - \gamma_j P_c\left(t\right)\alpha_j^*, \tag{1.8}$$

for $j = 0, 1$, is symmetric.

Condition (1.8) is equivalent to

$$\begin{bmatrix}\delta_j & \gamma_j\end{bmatrix}\Phi_0\begin{bmatrix}-\alpha_j & \beta_j\end{bmatrix}^* + \beta_j\alpha_j^* + \delta_j\gamma_j^*,$$

for $j = 0, 1$, is symmetric. This is true for every symplectic matrix Φ_0 if and only if it is true for every matrix Φ_0, even if it is not symplectic. Then one can easily prove the following proposition.

Proposition 1.4.

$$\begin{bmatrix}\delta_j & \gamma_j\end{bmatrix}\Phi_0\begin{bmatrix}-\alpha_j & \beta_j\end{bmatrix}^* + \beta_j\alpha_j^* + \delta_j\gamma_j^*,$$

for $j = 0, 1$, is symmetric for every symplectic matrix Φ_0, if and only if $\beta_j\alpha_j^* + \delta_j\gamma_j^*$ is symmetric and $\beta_j G\delta_j^* = 0$, $\beta_j G\gamma_j^* = 0$, $\delta_j G\alpha_j^* = 0$, $\gamma_j G\alpha_j^* = 0$, for $j = 0, 1$, and every antisymmetric matrix G.

With this assumption, equation (1.7) is equivalent to

$$\det\left(ad^{*}-bc^{*}\right)=0, \tag{1.9}$$

where

$$\begin{aligned} a &= \beta_0-\delta_0 Q_c\left(t\right)-\gamma_0 P_c\left(t\right)\\ d &= \alpha_1-\delta_1 Q_s\left(t\right)-\gamma_1 P_s\left(t\right)\\ b &= \alpha_0-\delta_0 Q_s\left(t\right)-\gamma_0 P_s\left(t\right)\\ c &= \beta_1-\delta_1 Q_c\left(t\right)-\gamma_1 P_c\left(t\right). \end{aligned}$$

It is then natural to consider a symplectic matrix Φ defined by

$$\Phi=\begin{bmatrix} Q_2 & Q_1\\ P_2 & P_1\end{bmatrix},$$

where $Q_2=R_0\left(ad^{*}-bc^{*}\right)R_1^{*}$, with $\det R_0\neq 0$, $\det R_1\neq 0$.

Then, formula (1.9) is equivalent to $\det Q_2=0$.

Notice that, if Φ is of the form

$$\Phi=L_0+L_1\Phi_0 L_2+L_3\Phi_0^{*}L_4, \tag{1.10}$$

then

$$\begin{aligned} &\left(L_0\right)_{11}=R_0\left(\beta_0\alpha_1^{*}-\alpha_0\beta_1^{*}+\delta_0\gamma_1^{*}-\gamma_0\delta_1^{*}\right)R_1^{*},\\ &\left(L_1\right)_{11}=R_0\delta_0,\quad \left(L_1\right)_{12}=R_0\gamma_0,\\ &\left(L_2\right)_{11}=-\alpha_1^{*}R_1^{*},\quad \left(L_2\right)_{21}=\beta_1^{*}R_1^{*},\\ &\left(L_3\right)_{11}=R_0\alpha_0,\quad \left(L_3\right)_{12}=-R_0\beta_0,\\ &\left(L_4\right)_{11}=\delta_1^{*}R_1^{*},\quad \left(L_4\right)_{21}=\gamma_1^{*}R_1^{*}. \end{aligned}$$

As $\alpha_0\beta_1^{*}+\delta_0\gamma_1^{*}=\beta_0\alpha_1^{*}+\gamma_0\delta_1^{*}$, one obtains

$$\left(L_0\right)_{11}=2R_0\left(\beta_0\alpha_1^{*}-\alpha_0\beta_1^{*}\right)R_1^{*}=2R_0\left(\delta_0\gamma_1^{*}-\gamma_0\delta_1^{*}\right)R_1^{*}.$$

The main problem here involved is to discover conditions over the matrices L_0, L_1, L_2, L_3 and L_4, so that Φ is symplectic for every symplectic matrix Φ_0. More generally, the problem is to discover conditions over Φ, with $Q_2=R_0\left(ad^{*}-bc^{*}\right)R_1^{*}$, such that Φ is symplectic for every symplectic matrix Φ_0. These questions can be completely solved in dimension one as it is done in the Appendix.

Let us take a look to simple cases in dimension greater than one.

Assume that $L_0=L_3=L_4=0$ and that L_1 and L_2 are both symplectic or antisymplectic. Then Φ is symplectic for every symplectic matrix Φ_0. The same happens, *mutatis mutandis*, when $L_0=L_1=L_2=0$.

The purpose of this article is to use the polar representation theorem in order to obtain results on the Sturm-Liouville problem.

2. A theorem on two parameters dependent symplectic matrices

In this section we prove a theorem that we shall need later and is a good introduction to the method we use in this article.

As before, let $\tau \in [0,t] \subset [0,T[$ and $\lambda \in]l_{-1}, l_1[\subset \mathbb{R}$. Consider the C^1 function $(\tau,\lambda) \mapsto \Phi(\tau,\lambda)$, where $\Phi(\tau,\lambda)$ is symplectic.

In the following we shall denote $\frac{\partial}{\partial\lambda}(\cdot) \equiv (\cdot)'$ the eigenvalue derivative, the derivative with respect to λ.

We define

$$M_1 = \dot{\Phi}\Phi^{-1}, \;\; S_1 = -JM_1.$$

and

$$M_2 = \Phi'\Phi^{-1}, \;\; S_2 = -JM_2.$$

Notice that, as Φ, M_j and S_j are both time and eigenvalue dependent, we shall use, as we did already before, the notations $\Phi \equiv \Phi(\tau) \equiv \Phi(\tau,\lambda)$, $M_j \equiv M_j(\tau) \equiv M_j(\tau,\lambda)$, $S_j \equiv S_j(\tau) \equiv S_j(\tau,\lambda)$, and so on $(j = 1,2)$. We also naturally denote

$$\Phi = \begin{bmatrix} Q_2 & Q_1 \\ P_2 & P_1 \end{bmatrix}, \quad S_j = \begin{bmatrix} A_j & B_j^* \\ B_j & C_j \end{bmatrix},$$

and assume that C_1 and C_2 are C^1 functions.

Let $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, $\epsilon = \epsilon_1\epsilon_2$.

Let $\tau_0 \geq 0$ and $\chi :]\tau_0, T[\to]l_{-1}, l_1[$ a continuous function, such that $\epsilon\chi$ is strictly decreasing and $\lim_{\tau\to T} \epsilon\chi(\tau) = \epsilon l_0 \geq \epsilon l_{-\epsilon}$ and $\lim_{\tau\to\tau_0} \chi(\tau) = l_\epsilon$.

Assume that

$$\det Q_2(\tau,\lambda) = 0 \Rightarrow \epsilon(\lambda - \chi(\tau)) > 0, \tag{2.1}$$

and that

$$\epsilon(\lambda - \chi(\tau)) > 0 \Rightarrow \{\epsilon_1 C_1(\tau,\lambda) > 0 \wedge \epsilon_2 C_2(\tau,\lambda) > 0\}. \tag{2.2}$$

Theorem 2.1. *Under Conditions* (2.1) *and* (2.2)*, equation*

$$\det Q_2(\tau,\lambda) = 0,$$

defines implicitly n sets of continuous functions $\tau \mapsto \lambda_{jk}(\tau)$, $(j = 1,2,\ldots,n)$, with the index $k \in \mathbb{Z}$ and bounded below. Some of these sets, or all, may be empty. In each nonempty set these functions have a natural order: $\epsilon\lambda_{jk}(\tau) < \epsilon\lambda_{j,k+1}(\tau) < \epsilon\lambda_{j,k+2}(\tau) < \cdots$.

Let $l \in]l_{-1}, l_1[$ and $t \in [0, T[$, and assume that $\det Q_2(t,l) \neq 0$. *Denote by μ_j the cardinal of the set $\{k \in \mathbb{N} : \epsilon(\lambda_{jk}(t) - l) < 0\}$ and let $\mu = \sum_{j=1}^{n} \mu_j$. Then, μ is the number of times, counting the multiplicities, that $Q_2(\tau, l)$ is singular, for $\tau < t$.*

Proof. As the proof for $\epsilon = -1$ is similar, suppose that $\epsilon = 1$. Define

$$\mathcal{D} = \{(\tau, \lambda) : \tau \in]\tau_0, T[\,, \lambda \in]l_{-1}, l_1[\,, \lambda > \chi(\tau)\}$$

From Theorem 1.1, one has that

$$Q_1(\tau, \lambda) = r(\tau, \lambda) \sin \varphi(\tau, \lambda),$$

$$Q_2(\tau, \lambda) = r(\tau, \lambda) \cos \varphi(\tau, \lambda),$$

where $r(\tau, \lambda)$, $\varphi(\tau, \lambda)$, for $(\tau, \lambda) \in \mathcal{D}$, are C^1 matrix-valued functions such that $\det r(\tau, \lambda) \neq 0$ and $\varphi(\tau, \lambda)$ is symmetric for every (τ, λ) and the eigenvalues of φ are C^1 functions of τ and λ. Denote $\varphi_1(\tau, \lambda), \ldots, \varphi_n(\tau, \lambda)$ such eigenvalues. Then $\epsilon_1 \dot{\varphi}_1(\tau, \lambda)$, $\ldots, \epsilon_1 \dot{\varphi}_n(\tau, \lambda)$ and $\epsilon_2 \varphi_1'(\tau, \lambda), \ldots, \epsilon_2 \varphi_n'(\tau, \lambda)$ are positive continuous functions, for $(\tau, \lambda) \in \mathcal{D}$. The matrix $Q_2(\tau, l)$, with $\tau < t$, is singular if, with $\lambda = l$,

$$\varphi_j(\tau, \lambda) = \frac{\pi}{2} + k\pi, \tag{2.3}$$

for some $j = 1, \ldots, n$ and $k \in \mathbb{Z}$.

Notice that $\varphi_j(\tau, \lambda) > \varphi_j(0, \lambda)$, so that the set of possible k either is empty or is bounded below.

Consider the sets Λ_{jk} defined by equation (2.3):

$$\Lambda_{jk} = \left\{(\tau, \lambda) \in \mathcal{D} : \varphi_j(\tau, \lambda) = \frac{\pi}{2} + k\pi\right\},$$

If one of the sets Λ_{jk} is not empty, then, locally, it defines a function $\lambda_{jk}(\tau)$, and

$$\frac{d\lambda_{jk}}{d\tau}(\tau) = -\frac{\partial \varphi_j}{\partial \tau}(\tau, \lambda_{jk}(\tau)) \left(\frac{\partial \varphi_j}{\partial \lambda}(\tau, \lambda_{jk}(\tau))\right)^{-1},$$

because $\epsilon_1 / \epsilon_2 = 1$.

Therefore, $\dot{\lambda}_{jk}(\tau) < 0$. Hence, the sets Λ_{jk} defined by (2.3) are totally ordered: $(\tau_1, \lambda_1) \succ (\tau_2, \lambda_2)$ if $\tau_1 > \tau_2$ and $\lambda_1 < \lambda_2$. Λ_{jk} has an infimum (t_{jk}, l_{jk}). The case $t_{jk} > 0$ and $l_{jk} < l_1$ can not happen from the implicit function theorem. The case $t_{jk} = 0$ and $l_{jk} < l_1$ is impossible as formula (2.1) makes clear. Hence, $t_{jk} \geq 0$ and $l_{jk} = l_1$.

Hence, λ_{jk} are C^1 functions $\lambda_{jk}(\tau) :]t_{jk}, T[\to \mathbb{R}$, such that

$$\lim_{\tau \to t_{jk}} \lambda_{jk}(\tau) = l_1 \ , \ \frac{d}{d\tau}\lambda_{jk}(\tau) < 0 \ , \ \varphi_j(\tau, \lambda_{jk}(\tau)) = \frac{\pi}{2} + k\pi.$$

We remark that, namely from (2.1), we have

$$\lambda_{j,k+1}(\tau) > \lambda_{jk}(\tau) > \chi(\tau) .$$

Hence, one has that the following three assertions are equivalent:

a) There is a $\tau < t$, such that $\lambda_{jk}(\tau) = l$.

b) There is a $\tau < t$, such that $\varphi_j(\tau, l) = \frac{\pi}{2} + k\pi$.

c) $\lambda_{jk}(t) < l$.

From this, the theorem follows. $\square$

3. Some formulas

As before, let $\tau \in [0, t] \subset [0, T[$ and $\lambda \in]l_{-1}, l_1[\subset \mathbb{R}$. Consider the C^1 function $(\tau, \lambda) \mapsto \Phi(\tau, \lambda)$, where $\Phi(\tau, \lambda)$ is symplectic. We define

$$M_1 = \dot{\Phi}\Phi^{-1}, \quad S_1 = -JM_1.$$

Notice that, as Φ, M_1 and S_1 are both time and eigenvalue dependent, we shall use, as we did already before, the notations $\Phi \equiv \Phi(\tau) \equiv \Phi(\tau, \lambda)$, $M_1 \equiv M_1(\tau) \equiv M_1(\tau, \lambda)$, $S_1 \equiv S_1(\tau) \equiv S_1(\tau, \lambda)$, and so on.

In the following we shall denote $\frac{\partial}{\partial \lambda}(\cdot) \equiv (\cdot)'$ the eigenvalue derivative, the derivative with respect to λ.

It is now natural to compute Φ' and $\Phi'\Phi^{-1} \equiv M_2$.

Deriving both members of $\dot{\Phi} = M_1\Phi$ in order to λ, one obtains

$$\dot{\Phi}' = M_1'\Phi + M_1\Phi'. \tag{3.1}$$

We shall use now the variations of parameters method. Write $\Phi' = \Phi K$, where K is both time and eigenvalue dependent: $K \equiv K(\tau, \lambda)$.

Let $K_0 = K(0, \lambda) \equiv K(0)$. As $K(0, \lambda) = \Phi^{-1}(0)\,\Phi'(0)$, and

$$\Phi(\tau) = \Phi(0) + \int_0^\tau M_1(\sigma)\,\Phi(\sigma)\,d\sigma,$$

one has

$$\Phi'(\tau) = (\Phi(0))' + \int_0^\tau (M_1(\sigma)\,\Phi(\sigma))'\,d\sigma.$$

Hence, $\Phi'(0) = (\Phi(0))'$ and $K_0 = K(0, \lambda) = \Phi^{-1}(0)\,(\Phi(0))'$.

On the other hand, one obtains

$$\dot{\Phi}' = \dot{\Phi}K + \Phi\dot{K} = M_1\Phi K + \Phi\dot{K} = M_1\Phi' + \Phi\dot{K}. \tag{3.2}$$

Comparing (3.1) with (3.2), one has

$$M_1' \Phi = \Phi \dot{K}.$$

From this one concludes that $\dot{K} = \Phi^{-1} M_1' \Phi$. Therefore

$$K(\tau) = K_0 + \int_0^\tau \Phi^{-1}(\sigma) M_1'(\sigma) \Phi(\sigma) \, d\sigma.$$

From now on we shall use the notations:

$$F(\tau, \sigma) = \Phi(\tau) \Phi^{-1}(\sigma), \quad F_0(\tau, \sigma) = \Phi_0(\tau) \Phi_0^{-1}(\sigma).$$

Then

$$\begin{aligned} M_2(\tau) &\equiv \Phi' \Phi^{-1} = \Phi K \Phi^{-1} \\ &= \Phi(\tau) \Phi^{-1}(0) (\Phi(0))' \Phi^{-1}(\tau) \\ &\quad + \int_0^\tau F(\tau, \sigma) M_1'(\sigma) \Phi(\sigma) F^{-1}(\tau, \sigma) \, d\sigma. \end{aligned}$$

Notice that, if V is any $2n \times 2n$ eigenvalue dependent matrix,

$$\begin{aligned} \int_0^\tau \Phi^{-1}(\sigma) V M_1(\sigma) \Phi(\sigma) \, d\sigma &= \int_0^\tau \Phi^{-1}(\sigma) V \dot{\Phi}(\sigma) \, d\sigma \\ &= \left[\Phi^{-1}(\sigma) V \Phi(\sigma)\right]_0^\tau + \int_0^\tau \Phi^{-1}(\sigma) M_1(\sigma) V \Phi(\sigma) \, d\sigma. \end{aligned}$$

Hence,

$$\begin{aligned} M_2(\tau) &= \Phi(\tau) \left(\Phi^{-1}(0) (\Phi(0))' + \left[\Phi^{-1}(\sigma) V \Phi(\sigma)\right]_0^\tau\right) \Phi^{-1}(\tau) \\ &\quad + \int_0^\tau F(\tau, \sigma) G_1 F^{-1}(\tau, \sigma) \, d\sigma, \end{aligned}$$

with

$$G_1 \equiv M_1'(\sigma) - V M_1(\sigma) + M_1(\sigma) V \tag{3.3}$$

or, equivalently,

$$\begin{aligned} M_2(\tau) &= V + \Phi(\tau) \Phi^{-1}(0) \left((\Phi(0))' - V \Phi(0)\right) \Phi^{-1}(\tau) + \\ &\quad + \int_0^\tau F(\tau, \sigma) G_1 F^{-1}(\tau, \sigma) \, d\sigma. \end{aligned}$$

Choosing

$$V = (\Phi(0))' \Phi^{-1}(0), \tag{3.4}$$

one has

$$M_2(\tau) = V + \int_0^\tau F(\tau, \sigma) G_1 F^{-1}(\tau, \sigma) \, d\sigma, \tag{3.5}$$

with V defined by (3.4) and G_1 defined by (3.3).

Equation (3.5) can be written

$$M_2(\tau) = (\Phi(0))' \Phi^{-1}(0) + \int_0^\tau F(\tau,\sigma) G_2 F^{-1}(\tau,\sigma) d\sigma,$$

with

$$G_2 \equiv \Phi(0) \left(\Phi^{-1}(0) M_1(\sigma) \Phi(0)\right)' \Phi^{-1}(0).$$

4. First remarkable case

Let us take

$$\Phi = L_1 \Phi_0 L_2,$$

where

$$\dot{\Phi}_0 = M_0 \Phi_0, \quad M_0 = -JS_0.$$

L_1 and L_2 are both symplectic or both antisymplectic and eigenvalue dependent: $L_1 \equiv L_1(\lambda)$, $L_2 \equiv L_2(\lambda)$. As before, Φ, Φ_0, M_0 and S_0 are both time and eigenvalue dependent: $\Phi \equiv \Phi(\tau) \equiv \Phi(\tau,\lambda)$, $\Phi_0 \equiv \Phi_0(\tau) \equiv \Phi_0(\tau,\lambda)$, $M_0 \equiv M_0(\tau) \equiv M_0(\tau,\lambda)$, $S_0 \equiv S_0(\tau) \equiv S_0(\tau,\lambda)$ and so on.

As $\dot{\Phi} = L_1 \dot{\Phi}_0 L_2 = L_1 M_0 \Phi_0 L_2 = L_1 M_0 L_1^{-1} \Phi$, one has

$$M_1 = L_1 M_0 L_1^{-1},$$
$$K_0 = L_2^{-1} L_1^{-1} (L_1 L_2)'.$$

Then

$$M_2(\tau) = L_1 \Phi_0(\tau) L_1^{-1} (L_1 L_2)' L_2^{-1} \Phi_0^{-1}(\tau) L_1^{-1} + \int_0^\tau F(\tau,\sigma) M_1'(\sigma) F^{-1}(\tau,\sigma) d\sigma,$$

and

$$M_2(\tau) = V + \int_0^\tau F(\tau,\sigma) G_3 F^{-1}(\tau,\sigma) d\sigma,$$

where

$$V = (L_1 L_2)' (L_1 L_2)^{-1},$$

and

$$G_3 \equiv M_1'(\sigma) - V M_1(\sigma) + M_1(\sigma) V.$$

One also has the formula

$$M_2(\tau) = V + \int_0^\tau L_1 F_0(\tau,\sigma) G_4 F_0^{-1}(\tau,\sigma) L_1^{-1} d\sigma, \tag{4.1}$$

where

$$G_4 \equiv M_0' + M_0 L_2' L_2^{-1} - L_2' L_2^{-1} M_0.$$

Remark 4.1. If $(L_1)_{12} = 0$, $\det((L_1)_{11}) \neq 0$ and $C_0 > 0$ ($C_0 < 0$), then $C_1 = (L_1)_{11} C_0 (L_1)_{11}^* > 0$ (< 0).

4.1. Example: the Morse index theorem.
Let N a symmetric $n \times n$ matrix. Define $Q_1 = Q_s$ and $Q_2 = Q_c + Q_s N$. Then Q_1 and Q_2 are isotropic, $W = I$. Hence, from Theorem 1.1, one has that

$$Q_1(\tau) = Q_s(\tau) = r(\tau)\sin\varphi(\tau),$$

$$Q_2(\tau) = Q_c(\tau) + Q_s(\tau) N = r(\tau)\cos\varphi(\tau), \tag{4.2}$$

where $r(\tau)$, $\varphi(\tau)$, for $\tau \in [0, T[$, are C^1 matrix-valued functions such that $\det r(\tau) \neq 0$ and $\varphi(\tau)$ is symmetric for every τ and the eigenvalues of φ are C^1 functions of τ. Denote $\varphi_1(\tau), \dots, \varphi_n(\tau)$ such eigenvalues, with $\varphi_j(0) = 0$. Then $\dot\varphi_1(\tau), \dots, \dot\varphi_n(\tau)$ are positive continuous functions.

Let $t \in [0, T[$. Assume that $Q_2(t)$ is invertible and that $\varphi_j(0) = 0$, $j = 1, \dots, n$, and define $\mu_j \in \mathbb{Z}$, such that

$$-\frac{\pi}{2} + \mu_j \pi < \varphi_j(t) < \frac{\pi}{2} + \mu_j \pi.$$

Define the index μ:

$$\mu = \sum_{j=1}^{n} \mu_j. \tag{4.3}$$

Then, μ is the number of times that $Q_2(\tau)$ is singular, for $\tau \in [0, t]$, taking into account the multiplicity of the singularity, i.e. the dimension of $\ker Q_2$.

Consider now the Lagrangian

$$L(q, \dot q, \tau) = \frac{1}{2}\left(\dot q, C(\tau)^{-1}\dot q\right) - \left(\dot q, C(\tau)^{-1} B(\tau) q\right) - \frac{1}{2}\left(q, \mathcal{A}(\tau) q\right),$$

where $\mathcal{A} = A - B^* C^{-1} B$.

Consider now the real separable Hilbert space $\mathcal{H}$, whose elements are the continuous functions $\gamma : [0, t] \to \mathbb{R}^n$,

$$\gamma(\tau) = -\int_\tau^t \dot\gamma(\sigma)\, d\sigma,$$

for $\dot{\gamma} \in L^2([0,t];\mathbb{R}^n)$. The inner product $\langle .,. \rangle$ in $\mathcal{H}$ is defined by

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \left(\dot{\gamma}_1(\tau), C(\tau)^{-1} \dot{\gamma}_2(\tau) \right) d\tau.$$

One denotes $\langle \gamma, \gamma \rangle = \|\gamma\|^2$.

To the Lagrangian L corresponds the action

$$\mathcal{S}(\gamma) = \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), \tau) \, d\tau + \frac{1}{2} (\gamma(0), N\gamma(0)),$$

where N, as before, is a symmetric $n \times n$ matrix.

The quadratic form $\mathcal{S} : \mathcal{H} \to \mathbb{R}$, defines a symmetric operator $\mathcal{L}(t) \equiv \mathcal{L} : \mathcal{H} \to \mathcal{H}$, $\mathcal{S}(\gamma) = \frac{1}{2} \langle \gamma, \mathcal{L}\gamma \rangle$,

$$\begin{aligned}
\langle \gamma_1, \mathcal{L}\gamma_2 \rangle = & \int_0^t \left(\dot{\gamma}_1(\tau), C(\tau)^{-1} \dot{\gamma}_2(\tau) \right) d\tau \\
& - \int_0^t \left(\dot{\gamma}_1(\tau), C(\tau)^{-1} B(\tau) \gamma_2(\tau) \right) d\tau \\
& - \int_0^t \left(\dot{\gamma}_2(\tau), C(\tau)^{-1} B(\tau) \gamma_1(\tau) \right) d\tau \\
& - \int_0^t (\gamma_1(\tau), \mathcal{A}(\tau) \gamma_2(\tau)) \, d\tau + (\gamma_1(0), N\gamma_2(0)),
\end{aligned}$$

which has the following expression

$$\begin{aligned}
(\mathcal{L}\gamma)(\tau) = & \, \gamma(\tau) + \int_\tau^t B(\sigma) \gamma(\sigma) \, d\sigma \\
& - \int_\tau^t C(\sigma) \, d\sigma \int_0^\sigma B^*(\theta) C(\theta)^{-1} \dot{\gamma}(\theta) \, d\theta \\
& - \int_\tau^t C(\sigma) \, d\sigma \int_0^\sigma \mathcal{A}(\theta) \gamma(\theta) \, d\theta + \int_\tau^t C(\sigma) \, d\sigma N\gamma(0).
\end{aligned}$$

$\mathcal{L}$ is the sum of four symmetric operators. The first one is the identity. The second one, which involves B, is a Hilbert-Schmidt operator. The third one, which involves $\mathcal{A}$, is a trace class operator. The forth one, which involves N, is a finite rank operator.

The eigenvalues λ of $\mathcal{L}$ are given by the equation

$$\mathcal{L}\gamma = \lambda\gamma, \quad \gamma \in \mathcal{H}, \quad \gamma \neq 0. \tag{4.4}$$

Assume that $\lambda \neq 1$ and put $\varepsilon = (1-\lambda)^{-1}$. As $\frac{d\varepsilon}{d\lambda} = (1-\lambda)^{-2} > 0$, we shall use ε instead of λ as a parameter, and $(\cdot)' \equiv \frac{\partial}{\partial \varepsilon}(\cdot)$.

Then, one has

$$|\varepsilon| > \left(at + bt^2\right)^{-1}, \tag{4.5}$$

where $a, b > 0$ (see [5]).

Define

$$A_1 = \varepsilon A + \left(\varepsilon^2 - \varepsilon\right) B^* C^{-1} B = \varepsilon \mathcal{A} + \varepsilon^2 B^* C^{-1} B$$
$$B_1 = \varepsilon B, \quad C_1 = C.$$

Call $\mathcal{L}_\varepsilon$ the operator $\mathcal{L}$ where one puts A_1, B_1, C_1 and εN instead of A, B, C and N. Notice that $\mathcal{L} = \mathcal{L}_1$. Then equation (4.4) becomes

$$\mathcal{L}_\varepsilon \gamma = 0, \quad \gamma \in \mathcal{H}\ , \gamma \neq 0.$$

This equation can be rewritten

$$\dot{\gamma} = B_1 \gamma + C_1 \beta, \quad \dot{\beta} = -A_1 \gamma - B_1^* \beta,$$
$$\gamma(t) = 0, \quad \beta(0) - \varepsilon N \gamma(0) = 0.$$

Put $L_1 = I_{2n}$ and

$$L_2 = \begin{bmatrix} f I_n & k f I_n \\ \varepsilon f N & f^{-1} I_n + k \varepsilon f N \end{bmatrix},$$

where k is constant and $f \equiv f(\varepsilon) \neq 0$.

Then $\Phi = L_1 \Phi_0 L_2 = \Phi_0 L_2$. Put $\Phi_{11} = Q_{\varepsilon,2}$, $\Phi_{12} = Q_{\varepsilon,1}$ and so on. Hence, $Q_{\varepsilon,2} = f\left(Q_c + \varepsilon Q_s N\right)$ and $Q_2 = f^{-1} Q_{1,2}$.

Then $\left(L_2' L_2^{-1}\right)_{12} = 0$, and if $f + 2f'\varepsilon = 0$,

$$\left(L_2' L_2^{-1}\right)_{22} = -\left(L_2' L_2^{-1}\right)_{11} = (2\varepsilon)^{-1}, \quad \left(L_2' L_2^{-1}\right)_{21} = 0.$$

Now, one computes G_4:

$$M_0' + M_0 L_2' L_2^{-1} - L_2' L_2^{-1} M_0 = \begin{bmatrix} B & \varepsilon^{-1} C \\ -\varepsilon B^* C^{-1} B & -B^* \end{bmatrix}.$$

Denoting

$$\begin{bmatrix} X & Z \\ W & Y \end{bmatrix} = \Phi_0(\tau)\, \Phi_0^{-1}(\sigma),$$

one has

$$C_2 = \varepsilon^{-1} \int_0^\tau \left(XC - \varepsilon Z B^*\right) C^{-1} \left(C X^* - \varepsilon B Z^*\right) d\sigma.$$

Then $\varepsilon C_2 > 0$ for $\tau > 0$.

From this, from (4.5) and from Theorem 2.1 one can easily state the following theorem, whose complete proof can be seen in detail in [5].

Theorem 4.1. *Let $\lambda(t)$ be an eigenvalue of the operator $\mathcal{L}(t)$. Then, there are three possibilities: 1) $\lambda(t) = 1$; 2) (and 3)) $\lambda(t) > 1$ ($\lambda(t) < 1$); in this case there exists a $t_0 \geq 0$ and a continuous function $\lambda(\tau)$, for $\tau \in [t_0, t]$, such that $\lambda(\tau)$ is an eigenvalue of the operator $\mathcal{L}(\tau)$ and $\lambda(t_0) = 1$; moreover, $\lambda(\tau)$ is C^1 in $]t_0, t]$ with $\dot{\lambda}(\tau) > 0$ ($\dot{\lambda}(\tau) < 0$).*

The eigenvalues of $\mathcal{L}(t)$ which are different from 1 can be organized in $2n$ sets; n for those > 1, n for those < 1. Some of these sets may be empty. In each set, the eigenvalues have a natural order: $\lambda_0(\tau) > \lambda_1(\tau) > \cdots > 1$, or $\lambda_0(\tau) < \lambda_1(\tau) < \cdots < 1$, for every τ. In particular, the eigenspace of $\lambda \neq 1$ has at most dimension n.

Let $Q_2 \equiv Q_c + Q_s N$, be a solution of the system (1.1)*. Then, $Q_2(t)$ is invertible if and only if $\mathcal{L}(t)$ is invertible and the number of the negative eigenvalues of $\mathcal{L}$ (its Morse index) is μ, as defined by* (4.3).

4.2. Example.
Let $A_0 = (1-\mu)A_3 + \mu A_4$, $B_0 = (1-\mu)B_3 + \mu B_4$, $C_0 = (1-\mu)C_3 + \mu C_4$. Assume that A_3, A_4, B_3, B_4, C_3, C_4, L_1 and L_2 are μ-independent and that L_1 and L_2 are symplectic. We shall use μ instead of λ as a parameter, and $(\cdot)' \equiv \frac{\partial}{\partial\mu}(\cdot)$. Then

$$S_0' = \begin{bmatrix} A_4 - A_3 & B_4 - B_3 \\ B_4^* - B_3^* & C_4 - C_3 \end{bmatrix} \equiv \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}.$$

If

$$\begin{bmatrix} X(\tau,\sigma) & Z(\tau,\sigma) \\ W(\tau,\sigma) & Y(\tau,\sigma) \end{bmatrix} \equiv \begin{bmatrix} X & Z \\ W & Y \end{bmatrix} = L_1 \Phi_0(\tau)\Phi_0^{-1}(\sigma),$$

then

$$C_2 = \int_0^\tau \big(XC(\sigma)X^* + ZA(\sigma)Z^* - XB(\sigma)Z^* - ZB^*(\sigma)X^* \big)\, d\sigma.$$

Hence, if $JS_0'J \leq 0$, $\varphi(\tau,\mu_1) \leq \varphi(\tau,\mu_2)$ for $\mu_1 \leq \mu_2$ and we have proved the following theorem:

Theorem 4.2. *If $JS_0'J \leq 0$ ($JS_0'J \geq 0$), $\varphi(\tau,\mu)$ is an increasing (decreasing) function of μ for every τ. Moreover, if, for every τ, there exists $\sigma < \tau$ such that $(JS_0'J)(\sigma) < 0$ ($JS_0'J > 0$), then $\varphi(\tau,\mu)$ is a strictly increasing (decreasing) function of μ for every $\tau > 0$.*

Notice that if L_1 and L_2 are antisymplectic one has to reverse the inequalities involving $JS_0'J$ in this theorem.

5. Second remarkable case

Let us take

$$\Phi = L_1 \Phi_0^* L_2,$$

where

$$\dot{\Phi}_0 = M_0 \Phi_0, \;\; M_0 = -JS_0.$$

L_1 and L_2 are both symplectic or both antisymplectic and eigenvalue dependent: $L_1 \equiv L_1(\lambda)$, $L_2 \equiv L_2(\lambda)$. As before, Φ, Φ_0, M_0 and S_0 are both time and eigenvalue dependent: $\Phi \equiv \Phi(\tau) \equiv \Phi(\tau, \lambda)$, $\Phi_0 \equiv \Phi_0(\tau) \equiv \Phi_0(\tau, \lambda)$, $M_0 \equiv M_0(\tau) \equiv M_0(\tau, \lambda)$, $S_0 \equiv S_0(\tau) \equiv S_0(\tau, \lambda)$ and so on.

$$\begin{aligned} M_1 &= \dot{\Phi}\Phi^{-1} = L_1 \Phi_0^* M_0^* \Phi_0^{*-1} L_1^{-1} \\ &= \Phi L_2^{-1} M_0^* L_2 \Phi^{-1}. \\ M_2 &= \Phi' \Phi^{-1} = \left(L_1' \Phi_0^* L_2 + L_1 \Phi_0^{*\prime} L_2 + L_1 \Phi_0^* L_2'\right) L_2^{-1} \Phi_0^{*-1} L_1^{-1}. \\ M_2 &= \Phi' \Phi^{-1} = L_1' L_1^{-1} + L_1 \Phi_0^{*\prime} \Phi_0^{*-1} L_1^{-1} + L_1 \Phi_0^* L_2' L_2^{-1} \Phi_0^{*-1} L_1^{-1}. \end{aligned}$$

Notice that $\left(\Phi_0^{*\prime}\Phi_0^{*-1}\right)^* = \Phi_0^{-1}\Phi_0'$ is $K \equiv K(\tau, \lambda)$, as defined in this section when we replace Φ by Φ_0. In this situation, $K_0 = 0$ and M_1 is M_0.

$$K \equiv K(\tau) = \int_0^\tau \Phi_0^{-1}(\sigma)\, M_0'(\sigma)\, \Phi_0(\sigma)\, d\sigma.$$

Then

$$\begin{aligned} M_2(\tau) = {} & L_1' L_1^{-1} + L_1 \Phi_0^* L_2' L_2^{-1} \Phi_0^{*-1} L_1^{-1} \\ & + L_1 \left(\int_0^\tau \Phi_0^*(\sigma)\, M_0^{*\prime}(\sigma)\, \Phi_0^{*-1}(\sigma)\, d\sigma \right) L_1^{-1}. \\ M_2(\tau) = {} & L_1' L_1^{-1} + \Phi L_2^{-1} L_2' \Phi^{-1} \\ & + L_1 \left(\int_0^\tau \Phi_0^*(\sigma)\, M_0^{*\prime}(\sigma)\, \Phi_0^{*-1}(\sigma)\, d\sigma \right) L_1^{-1}. \end{aligned}$$

Theorem 5.1. *Let $(L_2)_{22} = 0$, $\det\big((L_2)_{12}\big) \neq 0$, $Q_2(\tau) = r(\tau)\cos\varphi(\tau)$ and $Q_1(\tau) = r(\tau)\sin\varphi(\tau)$. Denote $\varphi_1(\tau), \ldots, \varphi_n(\tau)$ the eigenvalues of $\varphi(\tau)$. Then, if $C_0 > 0$ ($C_0 < 0$) and $\sin\varphi_j(\tau_0) = 0$, then $\varphi_j(\tau)$ is decreasing (increasing) in a neighborhood of τ_0.*

Proof. Denote

$$\begin{aligned}
C_3 &= -(L_2)^*_{12} C_0 (L_2)_{12}, \\
B_3 &= -(L_2)^*_{12} C_0 (L_2)_{11} + (L_2)^*_{12} B_0 (L_2)_{21}, \\
A_3 &= -(L_2)^*_{11} C_0 (L_2)_{11} - (L_2)^*_{21} A_0 (L_2)_{21} \\
&\quad + (L_2)^*_{11} B_0 (L_2)_{21} + (L_2)^*_{21} B_0^* (L_2)_{11}.
\end{aligned}$$

Then

$$C_1 = Q_2 C_3 Q_2^* - Q_2 B_3 Q_1^* - Q_1 B_3^* Q_2^* + Q_1 A_3 Q_1^*.$$

Let $U \equiv U(\tau)$ a C^1 orthogonal matrix defined in a neighborhood of τ_0 and $\Phi = U^* \varphi U$. Then, as, for $k \geq 1$,

$$\mathcal{C}_\varphi^k \dot{\varphi} = U\big(-\mathcal{C}_\Phi^{k+1}(U^* \dot{U}) + \mathcal{C}_\Phi^k \dot{\Phi}\big) U^*,$$

from formula (1.6), one has

$$\frac{\sin \mathcal{C}_\Phi}{\mathcal{C}_\Phi} \dot{\Phi} - (\sin \mathcal{C}_\Phi)(U^* \dot{U}) = U^* r^{-1} C_1 r^{*-1} U.$$

One can choose U such that $\Phi(\tau_0)$ is diagonal and $\Phi = \mathrm{diag}(\Phi_1, \Phi_2)$, with $\sin \Phi_1(\tau_0) \neq 0$, $\sin \Phi_2(\tau_0) = 0$.

Then, one obtains:

$$\left(\frac{\sin \mathcal{C}_\Phi}{\mathcal{C}_\Phi} \dot{\Phi}\right)_{22} = \dot{\Phi}_2, \quad \big((\sin \mathcal{C}_\Phi)(U^* \dot{U})\big)_{22}(\tau_0) = 0,$$

and

$$\begin{aligned}
U^* r^{-1} C_1 r^{*-1} U &= \cos \Phi U C_3 U^* \cos \Phi - \cos \Phi U B_3 U^* \sin \Phi \\
&\quad - \sin \Phi U B_3^* U^* \cos \Phi + \sin \Phi U A_3 U^* \sin \Phi.
\end{aligned}$$

Hence

$$\big(U^* r^{-1} C_1 r^{*-1} U\big)_{22}(\tau_0) = (\cos \Phi U C_3 U^* \cos \Phi)_{22}(\tau_0) < 0.$$

and

$$\dot{\Phi}_2(\tau_0) = (\cos \Phi U C_3 U^* \cos \Phi)_{22}(\tau_0) < 0.$$

Then $\dot{\Phi}_2(\tau) < 0$ in a neighborhood of τ_0 and the theorem follows. □

Similarly one can prove the following theorem:

Theorem 5.2. *Let $(L_2)_{21} = 0$, $\det((L_2)_{11}) \neq 0$, $Q_2(\tau) = r(\tau)\cos\varphi(\tau)$ and $Q_1(\tau) = r(\tau)\sin\varphi(\tau)$. Denote $\varphi_1(\tau), \ldots, \varphi_n(\tau)$ the eigenvalues of $\varphi(\tau)$. Then, if $C_0 > 0$ ($C_0 < 0$) and $\cos\varphi_j(\tau_0) = 0$, then $\varphi_j(\tau)$ is decreasing (increasing) in a neighborhood of τ_0.*

5.1. Example.
Let $A_0 = (1-\mu)A_3 + \mu A_4$, $B_0 = (1-\mu)B_3 + \mu B_4$, $C_0 = (1-\mu)C_3 + \mu C_4$. Assume that A_3, A_4, B_3, B_4, C_3 and C_4 are μ-independent. We shall use μ instead of λ as a parameter, and $(\cdot)' \equiv \frac{\partial}{\partial \mu}(\cdot)$. Then

$$S_0' = \begin{bmatrix} A_4 - A_3 & B_4 - B_3 \\ B_4^* - B_3^* & C_4 - C_3 \end{bmatrix} \equiv \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}.$$

Define

$$L_1 = \begin{bmatrix} \alpha_0 & -\beta_0 \\ \beta_0 & \alpha_0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} (1-\mu)\delta_3 + \mu\delta_4 & -I_n \\ I_n & 0 \end{bmatrix}$$

with $(\alpha_0\alpha_0^* + \beta_0\beta_0^*)^{-1/2} = I_n$, $\alpha_0\beta_0^* = \beta_0\alpha_0^*$, $\delta_3 = \delta_3^*$ and $\delta_4 = \delta_4^*$.

If

$$\begin{bmatrix} X(\tau) & Z(\tau) \\ W(\tau) & Y(\tau) \end{bmatrix} \equiv \begin{bmatrix} X & Z \\ W & Y \end{bmatrix} = L_1\Phi_0^*(\tau),$$

then

$$(5.1) \quad \begin{aligned} C_2 &= Q_1(\delta_4 - \delta_3)Q_1^* \\ &\quad - \int_0^\tau (ZC(\sigma)Z^* + XA(\sigma)X^* + XB^*(\sigma)Z^* + ZB(\sigma)X^*)\, d\sigma. \end{aligned}$$

Hence, if $S_0' \le 0$ and $\delta_4 - \delta_3 \ge 0$, $\varphi(\tau,\mu_1) \le \varphi(\tau,\mu_2)$ for $\mu_1 \le \mu_2$ and we have proved the following theorem:

Theorem 5.3. *If $S_0' \le 0$ and $\delta_4 - \delta_3 \ge 0$, $\varphi(\tau,\mu)$ is an increasing function of μ for every τ. Moreover, if $\delta_4 - \delta_3 > 0$ or, for every τ, there exists $\sigma < \tau$ such that $(S_0')(\sigma) < 0$, then $\varphi(\tau,\mu)$ is a strictly increasing function of μ for every $\tau > 0$.*

5.2. Example: the Sturm-Liouville problem.
Consider the Sturm-Liouville equation

$$(5.2) \qquad \left(C_0^{-1}\dot{q}\right)^{\cdot} + (-D + \lambda E)\, q = 0,$$

subject to the separated end conditions

$$(5.3) \qquad \begin{aligned} \beta_0 q(0) + \alpha_0 \left(C_0^{-1}\dot{q}\right)(0) = 0 \\ \delta_1 q(t) + \gamma_1 \left(C_0^{-1}\dot{q}\right)(t) = 0. \end{aligned}$$

In this case $A_0 = -D + \lambda E$, $B_0 = 0$; C_0, D and E are τ dependent and λ independent; $C_0, E > 0$. The matrices α_0, β_0, γ_1, δ_1 are λ independent. In this case $\beta_1 = \alpha_1 = \delta_0 = \gamma_0 = 0$One also has

$$\alpha_0\beta_0^* = \beta_0\alpha_0^*, \quad \gamma_1\delta_1^* = \delta_1\gamma_1^*.$$

Assume also that $\alpha_0\alpha_0^* + \beta_0\beta_0^* > 0$, $\det\gamma_1 \neq 0$. It is clear that one can replace δ_1 by $\gamma_1^{-1}\delta_1 \equiv \delta$ (a symmetric matrix) and γ_1 by I_n. One can also replace α_0 by $(\alpha_0\alpha_0^* + \beta_0\beta_0^*)^{-1/2}\,\alpha_0$ and β_0 by $(\alpha_0\alpha_0^* + \beta_0\beta_0^*)^{-1/2}\,\beta_0$ and have $\alpha_0\alpha_0^*+\beta_0\beta_0^* = I_n$, as we shall assume from now on. Then condition (1.7) is

$$\det\left(\begin{bmatrix}\beta_0 & \alpha_0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ \delta & I_n\end{bmatrix}\begin{bmatrix}Q_c\,(t) & Q_s\,(t)\\ P_c\,(t) & P_s\,(t)\end{bmatrix}\right) = 0.$$

Defining

$$\begin{aligned} Q_2 &= (\alpha_0 Q_c^* - \beta_0 Q_s^*)\,\delta + \alpha_0 P_c^* - \beta_0 P_s^*,\\ Q_1 &= -\alpha_0 Q_c^* + \beta_0 Q_s^*, \end{aligned}$$

one has that condition (1.7) is $\det Q_2\,(t) = 0$.

From now on we shall use the notation

$$Q_1 = r(\tau,\lambda)\sin\varphi(\tau,\lambda), \quad Q_2 = r(\tau,\lambda)\cos\varphi(\tau,\lambda).$$

Notice that the continuity condition on $\varphi\,(\tau,\lambda)$ implies that $\lambda \mapsto \varphi\,(0,\lambda)$ is constant.

We define $\Phi = L_1\Phi_0^* L_2$, Φ as in formula (1.2) and

$$L_1 = \begin{bmatrix}\alpha_0 & -\beta_0\\ \beta_0 & \alpha_0\end{bmatrix}, \quad L_2 = \begin{bmatrix}\delta & -I_n\\ I_n & 0\end{bmatrix}.$$

Then, if

$$\begin{bmatrix}X(\tau) & Z(\tau)\\ W(\tau) & Y(\tau)\end{bmatrix} \equiv \begin{bmatrix}X & Z\\ W & Y\end{bmatrix} = L_1\Phi_0^*(\tau),$$

we have

$$\begin{aligned} X &\equiv X(\tau) = \alpha_0 Q_c^*(\tau) - \beta_0 Q_s^*(\tau) = -Q_1\\ Z &\equiv Z(\tau) = \alpha_0 P_c^*(\tau) - \beta_0 P_s^*(\tau)\\ C_1 &= -ZC_0Z^* - XA_0X^*\\ M_2 &= \int_0^\tau \begin{bmatrix}X & Z\\ W & Y\end{bmatrix}\begin{bmatrix}0 & -E\\ 0 & 0\end{bmatrix}\begin{bmatrix}Y^* & -Z^*\\ -W^* & X^*\end{bmatrix} d\sigma \end{aligned}$$

$$C_2 = -\int_0^\tau X(\sigma)\,E(\sigma)\,X^*(\sigma)\,d\sigma. \tag{5.4}$$

We remark that $C_2 < 0$, for $\tau \in\,]0,t]$.

Lemma 5.4. *Consider the simpler case where $C_0 = cI_n$, $D = dI_n$, $E = eI_n$, $\delta = \theta I_n$ with $c, d, e, \theta \in \mathbb{R}$, $c, e > 0$. Then, there exists a symmetric matrix φ^- such that, for every $\tau \in]0, t]$,*

$$\lim_{\lambda \to +\infty} \varphi(\tau, \lambda) = -\infty, \quad \lim_{\lambda \to -\infty} \varphi(\tau, \lambda) = \varphi^-,$$

where $\tan \varphi^- = 0$. Moreover, φ^- is constant for $\tau \in]0, t]$.

Proof. Consider first $\lambda > d/e$. Define $\omega = \sqrt{c\,(-d + \lambda e)}$. Then

$$\begin{aligned} Q_2 &= \theta \big((\cos \omega\tau)\, \alpha_0 - c\omega^{-1} (\sin \omega\tau)\, \beta_0 \big) \\ &\quad - (\cos \omega\tau)\, \beta_0 - c^{-1}\omega\, (\sin \omega\tau)\, \alpha_0, \\ Q_1 &= -(\cos \omega\tau)\, \alpha_0 + c\omega^{-1} (\sin \omega\tau)\, \beta_0. \end{aligned}$$

Defining ψ and ρ, $\det \rho \neq 0$, such that

$$\alpha_0 = \rho \cos \psi, \quad c\omega^{-1}\beta_0 = \rho \sin \psi,$$

one has

$$\begin{aligned} Q_2 &= \rho \big(\theta \cos(\omega\tau I_n + \psi) - c^{-1}\omega \sin(\omega\tau I_n + \psi) \big), \\ Q_1 &= -\rho \cos(\omega\tau I_n + \psi). \end{aligned}$$

Then $Q_1^{-1} Q_2 = -\theta + c\omega^{-1} \tan(\omega\tau I_n + \psi)$, for every τ such that $\det \cos(\omega\tau I_n + \psi) \neq 0$.

Hence

$$\begin{aligned} Q_1 &= \rho\tilde{\rho} \sin \zeta(-\theta, c\omega^{-1}, \omega\tau I_n + \psi), \\ Q_2 &= \rho\tilde{\rho} \cos \zeta(-\theta, c\omega^{-1}, \omega\tau I_n + \psi), \end{aligned}$$

with ζ defined by (1.5) and

$$\tilde{\rho} = \sqrt{\cos^2(\omega\tau I_n + \psi) + \big(\theta \cos(\omega\tau I_n + \psi) - c^{-1}\omega \sin(\omega\tau I_n + \psi) \big)^2}.$$

As $Q_1 = r \sin \varphi$, $Q_2 = r \cos \varphi$, one has

$$r = \rho\tilde{\rho}, \quad \varphi = \zeta(-\theta, c\omega^{-1}, \omega\tau I_n + \psi).$$

As

$$\lim_{\sigma \to +\infty} \zeta(-\theta, c\omega^{-1}, \sigma) = -\infty,$$

the first part of the lemma follows.

Consider now the case $\lambda < d/e$. Define $\omega = \sqrt{c(d - \lambda e)}$. Then

$$\begin{aligned} Q_2 &= \theta \big((\cosh \omega\tau)\, \alpha_0 - c\omega^{-1} (\sinh \omega\tau)\, \beta_0 \big) \\ &\quad - (\cosh \omega\tau)\, \beta_0 + c^{-1}\omega (\sinh \omega\tau)\, \alpha_0, \\ Q_1 &= -(\cosh \omega\tau)\, \alpha_0 + c\omega^{-1} (\sinh \omega\tau)\, \beta_0. \end{aligned}$$

Defining η and ϱ, $\det \varrho \neq 0$, such that

$$\alpha_0 = \varrho \cos \eta, \quad \beta_0 = \varrho \sin \eta,$$

Then

$$Q_2^{-1} Q_1 = \frac{-\cos \eta + c\omega^{-1}(\tanh \omega\tau) \sin \eta}{\left(\theta + c^{-1}\omega(\tanh \omega\tau)\right) \cos \eta - \left(\theta c\omega^{-1}(\tanh \omega\tau) + 1\right) \sin \eta}$$

Hence, for every $\tau \in]0, t]$, there exists a λ_* such that, for $\lambda \leq \lambda_*$,

$$\left\| Q_2^{-1} Q_1 \right\| \leq \left(-|\theta| + c^{-1}\omega \left(\tanh \omega\tau\right) \right)^{-1},$$

and

$$\lim_{\lambda \to -\infty} \left\| Q_2^{-1} Q_1 \right\| = 0$$

For $\tau_* > 0$, this convergence is uniform in $[\tau_*, t]$. From this, the last part of the lemma follows. □

Theorem 5.5. *Consider the general case for C_0, D, E and δ. Then, for every $\tau \in]0, t]$,*

$$\lim_{\lambda \to +\infty} \varphi(\tau, \lambda) = -\infty, \qquad \lim_{\lambda \to -\infty} \tan \varphi(\tau, \lambda) = 0,$$

and $\varphi(\tau, \lambda)$ is a strictly decreasing function of λ.

Moreover, the eigenvalues of $\varphi(\tau, \lambda)$ converge to constant functions on $]0, t]$, as $\lambda \to -\infty$.

Proof. As C_2, defined by formula (5.4), is < 0, $\varphi(\tau, \lambda)$ is a strictly decreasing function of λ, for every $\tau \in]0, t]$.

For $\lambda > 0$, choose $\theta > \|\delta\|$, $d \geq D$, $0 < e \leq E$, $0 < c \leq C_0$, with $\theta, d, e, c \in \mathbb{R}$.

We use now Theorem 5.3. Put $\delta_3 = \delta$, $\delta_4 = \theta I_n$, $A_3 = -D + \lambda E$, $A_4 = (-d + \lambda e) I_n$, $C_3 = C_0$, $C_4 = cI_n$.

Then, from Theorem 5.3, one concludes that

$$\varphi(\tau, \lambda) \equiv \varphi(\tau, \lambda, 0) < \varphi(\tau, \lambda, 1),$$

and the first formula of the theorem is proved.

For $\lambda < 0$, choose $\theta > \|\delta\|$, $d \geq D$, $e \geq E$, $0 < c \leq C_0$, with $\theta, d, e, c \in \mathbb{R}$.

We use again Theorem 5.3. Put $\delta_3 = \delta$, $\delta_4 = \theta I_n$, $A_3 = -D + \lambda E$, $A_4 = (-d + \lambda e) I_n$, $C_3 = C_0$, $C_4 = cI_n$.

Then, from Theorem 5.3, one concludes that

$$\varphi_1(\tau, \lambda, 0) \equiv \varphi(\tau, \lambda) \equiv \varphi(\tau, \lambda, 0) < \varphi(\tau, \lambda, 1) \equiv \varphi_1(\tau, \lambda, 1),$$

the eigenvalues of $\varphi(\tau, \lambda)$ are bounded as $\lambda \to -\infty$.

For $\lambda < 0$, choose $\theta > \|\delta\|$, $d \leq D$, $0 < e \leq E$, $c \geq C_0$, with $\theta, d, e, c \in \mathbb{R}$.

We use once more Theorem 5.3. Put $\delta_3 = \delta$, $\delta_4 = -\theta I_n$, $A_3 = -D + \lambda E$, $A_4 = (-d + \lambda e) I_n$, $C_3 = C_0$, $C_4 = c I_n$.

Then, from Theorem 5.3, one concludes that

$$\varphi_2(\tau, \lambda, 0) \equiv \varphi(\tau, \lambda) \equiv \varphi(\tau, \lambda, 0) > \varphi(\tau, \lambda, 1) \equiv \varphi_2(\tau, \lambda, 1).$$

Choose λ_* the minimum of the $\lambda < 0$ such that $\det \cos \varphi_1(\tau, \lambda, \mu) = 0$ or $\det \cos \varphi_1(\tau, \lambda, \mu) = 0$, with $\mu \in [0, 1]$. It is clear that there exists such a λ_*, as φ_1 and φ_2 are bounded near $\lambda = -\infty$. Then, for $\lambda < \lambda_*$ and $\mu \in [0, 1]$, $\det \cos \varphi_1(\tau, \lambda, \mu) \neq 0$, $\det \cos \varphi_2(\tau, \lambda, \mu) \neq 0$. Hence, $\det Q_2(\tau, \lambda, \mu) \neq 0$ in both cases.

As, from (1.3) and (5.1), $\frac{d}{d\mu} Q_2^{-1} Q_1 > 0$ in the first case and < 0 in the second one, one obtains that, for $\lambda < \lambda_*$,

$$\tan \varphi_2(\tau, \lambda, 1) < Q_2^{-1} Q_1 < \tan \varphi_1(\tau, \lambda, 1).$$

Therefore

$$\left\| Q_2^{-1} Q_1 \right\| < \max \left\{ \left\| \tan \varphi_1(\tau, \lambda, 1) \right\|, \left\| \tan \varphi_2(\tau, \lambda, 1) \right\| \right\}.$$

From Theorem 5.3, one concludes that

$$\lim_{\lambda \to -\infty} \left\| Q_2^{-1} Q_1 \right\| = 0.$$

Then, for $\tau > 0$,

$$\lim_{\lambda \to -\infty} \tan \varphi_1(\tau, \lambda, \mu) = 0 \quad \text{and} \quad \lim_{\lambda \to -\infty} \tan \varphi_2(\tau, \lambda, \mu) = 0.$$

As $\lim_{\lambda \to -\infty} \varphi_1(\tau, \lambda, 1)$ and $\lim_{\lambda \to -\infty} \varphi_2(\tau, \lambda, 1)$ are constant in $]0, t]$, and the eigenvalues of these limit functions are integer multiple of π, the continuity of the functions φ_1 and φ_2 implies the last part of the theorem. □

Finally we have the following theorem:

Theorem 5.6. *For the Sturm-Liouville equation* (5.2)*, subject to conditions* (5.3)*, there are an infinite number of eigenvalues* $\lambda_{j,0} < \lambda_{j,1} < \lambda_{j,2} < \cdots < \lambda_{j,k} < \cdots$*,* $j = 1, 2, \ldots, n$*, with* $\lim_{k \to \infty} \lambda_{j,k} = +\infty$.

The eigenfunctions can be described as follows. There exists a matrix function $Q_1(\tau, \lambda) = r(\tau, \lambda) \sin \varphi(\tau, \lambda)$*, such that* $\det r(\tau, \lambda) \neq 0$ *and* $\varphi(\tau, \lambda)$ *is symmetric. The matrix functions* r *and* φ *are continuous. Consider the* φ *eigenvalues* $\varphi_j(\tau, \lambda)$ *and eigenvectors* $e_j(t, \lambda_{j,k})$*. Then the eigenfunction corresponding to* $\lambda_{j,k}$ *is* $Q_1(\tau, \lambda_{j,k}) e_j(t, \lambda_{j,k})$ *and* $\sin \varphi_j(\tau, \lambda_{j,k})$ *has exactly* k *zeros on* $]0, t[$.

Proof. Consider $\varphi(\tau,\lambda)$ and its eigenvalues $\varphi_j(\tau,\lambda)$, $j = 1,2, \ldots, n$. Then, from Theorem 5.5, $\varphi_j(\tau,\lambda)$ is strictly decreasing in λ, $\lim_{\lambda\to+\infty}\varphi_j(\tau,\lambda) = -\infty$, and there exists $l_j \in \mathbb{Z}$, such that $\lim_{\lambda\to-\infty}\varphi_j(\tau,\lambda) = l_j\pi$, for $\tau \in]0,t]$.

From Theorem 5.1, whenever $\varphi_j(\tau_l,\lambda) = l\pi$, for some $\tau_l \in]0,t[$, then $\varphi_j(\tau,\lambda)$ is a decreasing function of τ in a neighborhood of τ_l. Then, $\varphi_j(\tau,\lambda) < l\pi$ for $\tau > \tau_l$ and $\varphi_j(\tau,\lambda) > l\pi$ for $\tau < \tau_l$.

Clearly there exists a $\lambda_{j,k}$ such that $\varphi_j(t,\lambda_{j,k}) = \left(l_j - k - \frac{1}{2}\right)\pi$, for $k = 0,1,2,\ldots$.

For $\tau_* > 0$, there exists λ_* such that $\varphi_j(\tau_*,\lambda_*) = (l_j - 1)\,\pi$. Hence, for $\tau < \tau_*$, $\varphi_j(\tau_*,\lambda_*) > (l_j - 1)\,\pi$. Therefore $\varphi_j(0,\lambda_*) > (l_j - 1)\,\pi$. As $\lambda \mapsto \varphi_j(0,\lambda)$ is constant, it follows that $\varphi_j(0,\lambda) > (l_j - 1)\,\pi$ for every λ.

Define τ_m, $m = 1,2,\ldots,k$, $\varphi_j(\tau_m,\lambda_{j,k}) = (l_j - m)\,\pi$. The points τ_m are the unique points where $\sin\varphi_j(\tau,\lambda_{j,k}) = 0$ for $\tau \in]0,t]$. $\square$

Acknowledgements

The Mathematical Physics Group is supported by the portuguese Foundation for Science and Technology (FCT)

References

[1] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

[2] W. Kratz, "An oscillation theorem for self-adjoint differential systems and the Rayleigh principle for quadratic functionals", *J. London Math. Soc. (2)* **51** (1995), 401–416.

[3] J. Rezende, "A theorem on some linear Hamiltonian systems", *Differ. Equ. Dyn. Syst.* **5** (1997), 163–173.

[4] J. Rezende, "Time-dependent linear Hamiltonian systems and quantum mechanics", *Lett. Math. Phys.* **38** (1996), 117–127.

[5] J. Rezende, "A polar representation and the Morse index theorem", *Lett. Math. Phys.* **50** (1999), 91–102.

Appendix A.

Proposition A.1. *Let $n = 1$. $L_0 + L_1\Phi L_2$ is symplectic for every symplectic matrix Φ is equivalent to* $(\det L_0) + (\det L_1)\,(\det L_2) = 1$ *and* $L_1^* J L_0 J L_2^* = 0$.

If $L_0 + L_1\Phi L_2$ is symplectic for every symplectic matrix Φ, one of the following situations happens

a) L_0 is symplectic and $\det L_1 = \det L_2 = 0$, *with $L_1 \neq 0$ and $L_2 \neq 0$.*

b) L_0 *is symplectic and* $L_1 = 0$ *or* $L_2 = 0$.
c) $L_0 = 0$ *and* $\det L_1 \det L_2 = 1$.

Proof.

$$\begin{aligned}
&(L_0 + L_1\Phi L_2)J(L_0^* + L_2^*\Phi^* L_1^*)\\
&= L_0JL_0^* + L_0JL_2^*\Phi^*L_1^* + L_1\Phi L_2JL_0^* + L_1\Phi L_2JL_2^*\Phi^*L_1^*\\
&= (\det L_0)J + L_0JL_2^*\Phi^*L_1^* + L_1\Phi L_2JL_0^* + (\det L_1)(\det L_2)J = J.
\end{aligned}$$

As this must be true for Φ and $-\Phi$, one has

$$\begin{aligned}
(\det L_0) + (\det L_1)(\det L_2) &= 1,\\
L_0JL_2^*\Phi^*L_1^* + L_1\Phi L_2JL_0^* &= 0.
\end{aligned}$$

Hence, $L_1\Phi L_2JL_0^*$ is symmetric, for every symplectic matrix Φ. As $L_1(\Phi_1 + \Phi_2)L_2JL_0^*$ is also symmetric for any two symplectic matrices, $L_1\Phi L_2JL_0^*$ is symmetric even if Φ is not symplectic. As $K_1\Phi K_2$ is symmetric for every matrix Φ if and only if $K_2JK_1 = 0$, one easily concludes that $L_1^*JL_0JL_2^* = 0$. The proposition follows now without problems. □

Let $n = 1$ and $f_{11}, f_{12}, f_{21}, f_{22} : \mathbb{R}^4 \to \mathbb{R}$ four affine functions. Then, if

$$L = \begin{bmatrix} f_{11}(\Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}) & f_{12}(\Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}) \\ f_{21}(\Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}) & f_{22}(\Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}) \end{bmatrix}$$

is symplectic for every symplectic matrix Φ, one has that L is one of the forms

$$L = L_0 + L_1\Phi L_2, \quad L = L_0 + L_1\Phi^* L_2.$$

This can be proved by an explicit, and tedious, computation.

Notice that, following the proposition L_0 is either 0 or symplectic. If $L_0 = 0$, then L_1 and L_2 can be chosen such that $|\det L_1| = |\det L_2| = 1$, $(\det L_1)(\det L_2) = 1$. In this case they are either both symplectic or both antisymplectic.

In our problem $\Phi_{11} \equiv Q_c(t) = Q_c^*(t)$, $\Phi_{12} \equiv Q_s(t) = Q_s^*(t)$, $\Phi_{21} \equiv P_c(t) = P_c^*(t)$, $\Phi_{22} \equiv P_s(t) = P_s^*(t)$. Hence

$$f_{11}(\Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}) = x_0 + x_1\Phi_{11} + x_2\Phi_{12} + x_3\Phi_{21} + x_4\Phi_{22}$$

where

$$
\begin{aligned}
x_0 &= R(\beta_0\alpha_1 - \alpha_0\beta_1 + \delta_0\gamma_1 - \gamma_0\delta_1)\\
x_1 &= R(\alpha_0\delta_1 - \delta_0\alpha_1)\\
x_2 &= R(\delta_0\beta_1 - \beta_0\delta_1)\\
x_3 &= R(\alpha_0\gamma_1 - \gamma_0\alpha_1)\\
x_4 &= R(\gamma_0\beta_1 - \beta_0\gamma_1)
\end{aligned}
$$

where $R = R_0R_1$ is a real eigenvalue dependent parameter, $R \neq 0$.

Notice that $x_1x_4 - x_2x_3 = R^2(\delta_1\gamma_0 - \delta_0\gamma_1)(\beta_1\alpha_0 - \alpha_1\beta_0)$. As $x_0 = 2R(\beta_0\alpha_1 - \alpha_0\beta_1) = 2R(\delta_0\gamma_1 - \gamma_0\delta_1)$, one has that

$$x_1x_4 - x_2x_3 = 4^{-1}x_0^2.$$

Let $L_0 = I_2$, the 2×2 unit matrix. Then L can be of the following three forms:

a) $f_{11} = 1$, $f_{22} = 1$, $f_{12} = 0$;

b) $f_{11} = 1$, $f_{22} = 1$, $f_{21} = 0$;

c) there exists an $\kappa \neq 0$ such that $f_{22} - 1 = -(f_{11} - 1)$, $f_{12} = \kappa(f_{11} - 1)$, $f_{12} = -\kappa^{-1}(f_{11} - 1)$.

The case where L_0 is symplectic but $\neq I_2$ is easily derived from this one.

Let now $L_0 = 0$. Then $x_0 = 0$ and $x_1x_4 - x_2x_3 = 0$.

There are five possible situations: a) $x_1 \neq 0$, b) $x_1 = 0, x_4 \neq 0, x_3 = 0$, c) $x_1 = 0, x_4 \neq 0, x_2 = 0$, d) $x_1 = 0, x_4 = 0, x_2 = 0$, e) $x_1 = 0, x_4 = 0, x_3 = 0$.

	a)	b)	c)	d)	e)
$(L_1)_{11}$	a	$ax_2x_4^{-1}$	0	0	a
$(L_1)_{12}$	$ax_3x_1^{-1}$	a	a	a	0
$(L_1)_{21}$	b	$-\nu a^{-1}$ $+ bx_2x_4^{-1}$	$-\nu a^{-1}$	$-\nu a^{-1}$	b
$(L_1)_{22}$	$\nu a^{-1} + bx_3x_1^{-1}$	b	b	b	νa^{-1}
$(L_2)_{11}$	$a^{-1}x_1$	0	$a^{-1}x_3$	$a^{-1}x_3$	0
$(L_2)_{12}$	c	$-\nu ax_4^{-1}$	$-\nu ax_4^{-1}$ $+ cx_3x_4^{-1}$	c	$-\nu ax_2^{-1}$
$(L_2)_{21}$	$a^{-1}x_2$	$a^{-1}x_4$	$a^{-1}x_4$	0	$a^{-1}x_2$
$(L_2)_{22}$	$\nu ax_1^{-1} + cx_2x_1^{-1}$	c	c	νax_3^{-1}	c

where a, b and c are real eigenvalue dependent parameters, $a \neq 0$, and $\nu = \pm 1$; $\nu = 1$ in the symplectic case, $\nu = -1$ in the antisymplectic case.